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NON-DENTABLE SETS IN BANACH SPACES WITH SEPARABLE DUAL

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ABSTRACT

A non-RNP Banach space E is constructed such that E^* is separable and the RNP is equivalent to the PCP on the subsets of E.

The problem of the equivalence of the Radon–Nikodym Property (RNP) and the Krein–Milman Property (KMP) remains open for Banach spaces as well as for closed convex sets. A step forward has been made with Schachermayer's Theorem [S]. That result states that the two properties are equivalent on strongly regular sets. Rosenthal, [R], has shown that every non-RNP strongly regular closed convex set contains a non-dentable subset on which the norm and weak topologies coincide. In a previous paper ([A-D]) we proved that every non-RNP closed convex set contains a subset with a martingale coordinatization. Furthermore, we established the $P\alpha\ell$-representation for several cases. The remaining open case in the equivalence of the RNP and the KMP is that of B-spaces or closed convex sets with RNP equivalent to PCP on their subsets. A typical example of such a structure is $L^1[0,1]$. H. Rosenthal raised the question if this could occur in a space with separable dual. R. James ($[J_2]$) also posed a similar problem. The aim of the present paper is to give an example of a Banach space E with separable dual, failing the RNP, and such that the RNP is equivalent to the PCP on its subsets. As a consequence we get that E does not contain $c_0(\mathbb{N})$ isomorphically and hence it does not embed into a Banach space with an unconditional skipped F.D.D. On

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the other hand, E semiembeds into a Banach space with an unconditional basis. The last property allows us to conclude that every closed convex non-RNP subset of E contains a closed non-dentable set with a $P\alpha\ell$-representation. (We recall that a closed set K has a $P\alpha\ell$-representation if there is an affine, onto, one-to-one continuous map from the atomless probability measures on [0,1] to the set K.) In particular, the RNP is equivalent to the KMP on the subsets of E. The space E is realized by applying the Davis–Figiel–Johnson–Pelczynski factorization method to a convex symmetric set W of a Banach space E_u constructed in this paper. Finally, as a consequence of the methods used in the proofs of the example, we obtain that every separable B-space X such that X^{**}/X is isomorphic to $\ell^1(\Gamma)$ has the RNP.

We start with some definitions, notations and results, necessary for our constructions.

A closed convex bounded set K is said to be δ-non-dentable, $\delta > 0$, if every slice of K has diameter greater than δ. A closed convex set has the RNP if it contains no δ-non-dentable set. A closed subset K of a B-space has the P.C.P. if for every subset L of K and for all $\varepsilon > 0$ there exists a relatively weakly open neighbourhood of L with diameter less than ε. K is stongly regular if for every subset L of K and for every $\varepsilon > 0$, there exists a convex combination $\sum \lambda_i S_i$ of slices (S_i) of L, with $\mathrm{diam}(\sum \lambda_i S_i) < \varepsilon$. It is well known that the RNP implies the P.C.P., but the converse fails [B-R].

In the sequel $\mathcal{D}$ denotes the dyadic tree, namely the set of all finite sequences of the form $\alpha = (0, \varepsilon_1, ..., \varepsilon_n)$ with $\varepsilon_i = 0$ or 1. For α in $\mathcal{D}$, the length of α is denoted by $|\alpha|$. For $n \in N$, the set $\{\alpha \in \mathcal{D} : |\alpha| = n\}$ is called the n-th level of the tree $\mathcal{D}$. A natural order is induced on $\mathcal{D}$, that is $\alpha \prec \beta$ if the sequence α is an initial segment of the sequence β. Two elements α, β of $\mathcal{D}$ are called **incomparable** if they are incomparable in the above defined order. We note, for later use, that each α in $\mathcal{D}$ determines a unique basic clopen subset V_α in the Cantor's group $\{0,1\}^{\mathbb{N}}$ and α, β are incomparable iff $V_\alpha \cap V_\beta = \emptyset$.

A basic ingredient in the definition of the space E is Tsirelson's norm $||.||_T$ as it is defined in [F-J]. Let $(t_k)_{k=1}^{\infty}$ denote the canonical unit vector basis in c_{00}. For E, F finite non-void subsets of $\mathbb{N}$ we write '$E < F$' for '$\max E < \min F$'. For $x = \sum_{k=1}^{m} \lambda_k t_k$, Ex is $\sum_{k \in E} \lambda_k t_k$. The norm $||.||_T$ on Tsirelson's space T satisfies the following property.

For $x = \sum_{k=1}^{m} \lambda_k t_k$,

$$\|\sum_{k=1}^{m} \lambda_k t_k\|_T = \max\{\max_k |\lambda_k|, \frac{1}{2} \sup \sum_{j=1}^{n} \|E_j x\|_T\}$$

where the "sup" is taken over all choices

$$n \le E_1 < E_2 < \cdots < E_n$$

with $E_1, ..., E_n$ a sequence of intervals in the set of natural numbers. We recall that $(t_\kappa)_{\kappa=1}^{\infty}$ is an unconditonal basis for T , and T is a reflexive Banach space not containing any ℓ^p for $1 < p < \infty$.

The space E_U

The space E_u will be defined to have an unconditional basis indexed by the dyadic tree $\mathcal{D}$ and denoted by $(e_\alpha)_{\alpha\in\mathcal{D}}$. For a sequence of reals $(\lambda_\alpha)_{\alpha\in\mathcal{D}}$ which is eventually zero we define

$$\|\sum_{\alpha\in\mathcal{D}} \lambda_\alpha e_\alpha\| = \sup\{\|\sum_{i=1}^{\ell} \lambda_{\alpha_i} t_{k_i}\|_T : \ell \in N\,,\ \{\alpha_i\}_{i=1}^{\ell} \text{ are incomparable,}$$
$$|\alpha_i| = k_i,\ k_1 < k_2 < \cdots < k_\ell\}.$$

It is clear that $(e_\alpha)_{\alpha\in\mathcal{D}}$ is an unconditional basis for the space E_u defined by the above norm.

Next, we verify certain properties of the space E_u.

PROPOSITION 1: *The dual of the space E_u is separable.*

Proof: The space E_u has an unconditional basis, hence it is enough to show that ℓ^1 does not embed into E_u [J₁].

Suppose, on the contrary, that ℓ^1 embeds into E_u. Then, by standard arguments, we can find an increasing sequence of natural numbers $\ell_1 < \ell_2 < \cdots < \ell_k < \cdots$ and a normalized sequence $\{x_k\}_{k=1}^{\infty}$ in E_u, equivalent to the usual basis of ℓ^1, with

$$x_k = \sum_{\ell_k < |\alpha| \le \ell_{k+1}} \lambda_\alpha e_\alpha.$$

Now, for every choice of coefficients (μ_k),

$$\|\sum_{k=1}^{m}\mu_k x_k\| = \sup \|\sum_{k=1}^{m}\mu_k(\sum_i \lambda_{\alpha_i^k} t_{n_i^k})\|_T$$

where $n_i^k = |\alpha_i^k|$ and the "sup" is taken over all choices of sets $\{\alpha_i^k\}_{k,i}$ of incomparable α_i^k with $\ell_k < |\alpha_i^k| \leq \ell_{k+1}$ and $|\alpha_i^k|$ pairwise different.

By a known property of Tsirelson's norm (Lemma II.3 of [C-S]), we get

$$\|\sum_{k=1}^{m}\mu_k(\sum_i \lambda_{\alpha_i^k} t_{n_i^k})\|_T \leq 6\|\sum_{k=1}^{m}\mu_k t_{\ell_{k+1}}\|_T$$

for each such choice of $\{\alpha_i^k\}_{k,i}$. So

$$\|\sum_{k=1}^{m}\mu_k x_k\| \leq 6\|\sum_{k=1}^{m}\mu_k t_{\ell_{k+1}}\|_T$$

which gives that $\{t_{\ell_k}\}_{k=2}^{\infty}$ is equivalent to the basis of ℓ^1. This contradicts the reflexivity of T. ∎

A consequence of the above Proposition is that the basis $(e_\alpha)_{\alpha\in\mathcal{D}}$ is shrinking. Therefore, every x^{**} in E_u^{**} has a unique representation as

$$x^{**} = w^* - \lim_{n\to\infty} \sum_{|\alpha|\leq n} \lambda_\alpha e_\alpha := w^* - \sum_{\alpha\in\mathcal{D}} \lambda_\alpha e_\alpha$$

where $\lambda_\alpha = < x^{**}, e_\alpha^* >$.

We define the **support** of x^{**}, denoted by supp x^{**}, to be the set

$$\{\alpha \in \mathcal{D} : < x^{**}, e_\alpha^* > \neq 0\}.$$

LEMMA 2: *Let $x_1^{**}, \ldots, x_k^{**}$ in E_u^{**} be such that there are incomparable elements $\alpha_1, \ldots, \alpha_k$ in $\mathcal{D}$ so that* supp x_i^{**} *is contained in*

$$W_{\alpha_i} = \{\beta \in \mathcal{D} : \beta \prec \alpha_i \text{ or } \alpha_i \prec \beta\}.$$

Then,

$$d(x_1^{**} + \cdots + x_k^{**}, E_u) \geq \frac{1}{2}\sum_{i=1}^{k} d(x_i^{**}, E_u).$$

Proof: For $n < m$ we define

$$P_{[n,m]}(x^{**}) = \sum_{n\le|\alpha|\le m} \lambda_\alpha e_\alpha$$

and

$$P_{[n,\infty]}(x^{**}) = w^* - \sum_{n\le|\alpha|} \lambda_\alpha e_\alpha$$

where $\lambda_\alpha =< x^{**}, e^*_\alpha >$.

Using this notation, we have

$$d(x^{**}, E_u) = \lim_{n\to\infty} ||P_{[n,\infty]}(x^{**})||$$

and

$$||P_{[n,\infty]}(x^{**})|| = \lim_{m\to\infty} ||P_{[n,m]}(x^{**})||.$$

To establish the result it is enough to show that there exists n such that for all $m > n$

$$||P_{[m,\infty]}(\sum_{i=1}^k x_i^{**})|| \ge \frac{1}{2}\sum_{i=1}^k d(x_i^{**}, E_u).$$

Actually, $n = \max\{k, |\alpha_1|, ..., |\alpha_k|\}$.

Choose any $m > n$. We shall show that for all $\varepsilon > 0$,

$$||P_{[m,\infty]}(\sum_{i=1}^k x_i^{**})|| \ge \frac{1}{2}\sum_{i=1}^k d(x_i^{**}, E_u) - \varepsilon.$$

Given $\varepsilon > 0$, inductively we define $\{q_i, \ell_i\}_{i=1}^k$ such that

$$m < q_1 < \ell_1 < \cdots < q_k < \ell_k$$

and $||P_{[q_i,\ell_i]}(x_i^{**})|| > d(x_i^{**}, E_u) - \frac{\varepsilon}{k}$.

For each $1 \le i \le k$ there is a set $\{\beta^i_j : 1 \le j \le s(i)\}$ of incomparable elements of $\mathcal{D}$ which lie on different levels of $\mathcal{D}$, such that $q_i \le |\beta^i_j| \le \ell_i$ and

$$||P_{[q_i,\ell_i]}(x_i^{**})|| = ||\sum_{j=1}^{s(i)} \lambda_{\beta^i_j}\, t_{|\beta^i_j|}||_T.$$

Notice that $\alpha_i \prec \beta^i_j$ for all $j = 1, ..., s(i)$.

Observe that $\bigcup_{1\le i\le k}\{\beta^i_j : 1\le j\le s(i)\}$ consists of pairwise incomparable elements which lie on different levels of $\mathcal{D}$. So,

$$\begin{aligned}||P_{[m,\infty]}(\sum_{i=1}^{k} x_i^{**})|| &\geq ||P_{[m,\ell_k]}(\sum_{i=1}^{k} x_i^{**})|| \\ \geq ||\sum_{i=1}^{k}\sum_{j=1}^{s(i)} \lambda_{\beta^i_j}\, t_{|\beta^i_j|}||_T &\geq \frac{1}{2}\sum_{i=1}^{k} ||\sum_{j=1}^{s(i)} \lambda_{\beta^i_j}\, t_{|\beta^i_j|}||_T \\ = \frac{1}{2}\sum_{i=1}^{k} ||P_{[q_i,\ell_i]}(x_i^{**})|| &\geq \frac{1}{2}\sum_{i=1}^{k} d(x_i^{**},E_u)-\varepsilon. \qquad \blacksquare\end{aligned}$$

Consider the following closed convex subset of the unit ball of E_u:

$$K=\{x\in E_u : x=\sum_{n=0}^{\infty}\sum_{|\alpha|=n}\lambda_\alpha e_\alpha, \lambda_0=1, \lambda_\alpha\geq 0, \lambda_\alpha=\lambda_{(\alpha,0)}+\lambda_{(\alpha,1)}\}.$$

One can verify that K is the closed convex hull of the set $(d_\alpha)_{\alpha\in\mathcal{D}}$, where, for every α in $\mathcal{D}$, the vector d_α is defined by the conditions

$$e^*_\alpha(d_\alpha)=1,\quad e^*_{(\beta,0)}(d_\alpha)=e^*_{(\beta,1)}(d_\alpha)=\frac{1}{2}e^*_\beta(d_\alpha)\quad \text{and} \quad d_\alpha\in K.$$

It is easily checked that for every $\alpha\in\mathcal{D}$,

$$d_\alpha=\frac{1}{2}(d_{(\alpha,0)}+d_{(\alpha,1)}),\quad ||d_\alpha-d_{(\alpha,0)}||\geq\frac{1}{2}\quad \text{and} \quad ||d_\alpha-d_{(\alpha,1)}||\geq\frac{1}{2}$$

which means that $(d_\alpha)_{\alpha\in\mathcal{D}}$ is a $\frac{1}{2}$ tree. Consequently, K is non-dentable.

We set $W=\operatorname{co}(K\cup -K)$ and we denote by $\tilde{W}$ its w^*-closure in E_u^{**}. Notice that $x^{**}\in\tilde{W}$ iff $|e^*_\alpha(x^{**})|\leq 1$, and $e^*_{(\alpha,0)}(x^{**})+e^*_{(\alpha,1)}(x^{**})=e^*_\alpha(x^{**})$ for all α in $\mathcal{D}$. Hence, we can define a map T from the unit ball $M_1(\{0,1\}^N)$ of the space $M(\{0,1\}^N)$ to $\tilde{W}$

$$T: M_1(\{0,1\}^{\mathbb{N}})\to\tilde{W}$$

by the rule

$$T(\mu)=w^*-\sum_{\alpha\in\mathcal{D}}\mu(V_\alpha)e_a$$

where $V_\alpha=\{\gamma\in\{0,1\}^{\mathbb{N}} : \alpha \text{ is an initial segment of } \gamma\}$.

Clearly, T is one-to-one and onto. Furthermore,

$$||T(\mu)||\leq\sup\{\sum_{i=1}^{k}|\mu(V_{\alpha_i})| : \{\alpha_i\}_{i=1}^{k} \text{ incomparable}\}=||\mu||.$$

Hence, T is extended to a bounded linear operator from $M(\{0,1\}^{\mathbb{N}})$ onto the linear span of $\tilde{W}$ denoted by $<\tilde{W}>$.

The Space E

The space E is the result of the application of the Davis–Figiel–Johnson–Pelczynski factorization method to the set W defined above.

We give the precise definition and certain properties of the space E. For a detailed presentation, we refer the reader to [D-F-J-P]. In particular, P.1, P.2, and P.5, stated below, are established in Lemmata 2.1 and 3.1 of [D-F-J-P].

$$E = \{y \in E_u : |||y||| = (\sum_{n=1}^{\infty} ||y||_n^2)^{\frac{1}{2}} < \infty\}.$$

Here $||.||_n$ denotes the Minkowski's gauge of the set $2^n W + \frac{1}{2^n} B_{E_u}$.

Let $J : E \to E_u$ be the natural injection. The operator J is continuous. Furthermore, J satisfies the following properties.

P.1: *$J^{**} : E^{**} \to E_u^{**}$ is one-to-one and $J^{**}[E^{**}] \cap E_u = J[E]$.*

P.2: *J is a weak-to-weak homeomorphism on the bounded subsets of E.*

This is a consequence of P.1.

P.2 implies that $J[L]$ is closed for all closed convex bounded subsets L of E. In particular, J is a semiembedding.

P.3: *If L is a closed convex bounded subset of E failing the RNP, then $J[L]$ also fails the RNP.*

By P.2, $J[L]$ is closed. Suppose it has the RNP. Let S be a L-valued operator $S : L^1[0,1] \to E$; the operator JoS is representable by a function φ in $L^{\infty}_{J[L]}$. Then the function $\psi = J^{-1}\varphi$ represents the operator S. It follows that L has the RNP. (For more details we refer to [B-R].)

P.4a: *If L is a bounded subset of E and $J[L]$ fails the RNP, then L fails the RNP.*

P.4b: *If L is a bounded subset of E and $J[L]$ fails the P.C.P., then L fails the P.C.P.*

P.4c: *If L is a bounded subset of E and $J[L]$ is not strongly regular, then L is not strongly regular.*

P.4a,b,c follow from P.1. In particular, they are consequences of the fact that $J^*[E_u^*]$ is norm-dense in E^*.

P.5: *Let $\overline{<\tilde{W}>}$ denote the closed linear span of the w^*-closure $\tilde{W}$ of W in E_u^{**}. Then,*

$$J^{**}[E^{**}] \subseteq \overline{<\tilde{W}>}.$$

For this, notice that $B_{E^{**}} \subset 2^n\tilde{W} + \frac{1}{2^n}B_{E_u^{**}}$, hence

$$J^{**}[B_{E^{**}}] \subseteq \bigcap_n (2^n\tilde{W} + \frac{1}{2^n}B_{E_u^{**}}) \subseteq \overline{<\tilde{W}>}.$$

The following Proposition is an immediate consequence of the above properties.

PROPOSITION 3: *(i) The dual E^* of E is separable.*
(ii) The space E fails the RNP.

Proof: (i) Since, by P.1, J^{**} is one-to-one, $J^*[E_u^*]$ is norm-dense in E^*. Hence, by Proposition 1, E^* is separable.

(ii) Notice that $W \subseteq J[B_E]$. Since W fails the RNP, by P.4a we get that B_E fails the RNP.

We proceed now to the proof of the main property of the space E.

PROPOSITION 4: *Let C be a closed, convex, bounded, non-RNP subset of E. Then C fails the P.C.P.*

Proof: P.3 ensures that $J[C]$ is a non-RNP closed subset of E_u. Hence, for some $\delta > 0$, there exists a convex closed subset L of $J[C]$ which is δ-nondentable. Let $\tilde{L}$ denote the w^*-closure of L in E_u^{**}. We shall show the following:

(*) *For every choice $S_1, S_2, ..., S_n$ of slices of $\tilde{L}$ there exist x_i^{**} in S_i, $i = 1, 2, ..., n$, such that for all $(\lambda_i)_{i=1}^n \in \mathbb{R}_+^n$ with $\sum_{i=1}^n \lambda_i = 1$, we have*

$$d(\sum_{i=1}^n \lambda_i x_i^{**}, E_u) > \frac{\delta}{256}.$$

It follows from (*) that $J[C]$ is not strongly regular. By a result due to Bourgain [B] this yields that $J[C]$ fails the PCP. (See also [G-G-M-S].) By P.4a we then get that C fails the PCP.

Proof of (*): Let $S_1, S_2, ..., S_n$ be slices of $\tilde{L}$. Using Lemma 2.7 from [R], we choose for each $i = 1, ..., n$ an uncountable subset $(x_{\xi,i}^{**})_{\xi<\omega_1}$ of S_i such that

$$d(x_{\xi,i}^{**} - x_{\zeta,i}^{**}, E_u) > \frac{3\delta}{8} \quad \text{for } \xi \neq \zeta.$$

Recall that $\tilde{L}$ is a subset of $J^{**}[E^{**}] \subset \overline{<\tilde{W}>}$ and that $T[M(\{0,1\}^{\mathbb{N}})]$ is norm-dense in $\overline{<\tilde{W}>}$. Hence, there are $(\mu_{\xi,i})_{\xi<\omega_1, i\leq n}$ such that

$$||T\mu_{\xi,i} - x_{\xi,i}^{**}|| < \frac{\delta}{256}.$$

It is known ([L]) that $M(\{0,1\}^{\mathbb{N}}) = (\sum_{\gamma<2^\omega} \bigoplus L^1(\lambda_\gamma))_1$ where $\{\lambda_\gamma\}_{\gamma<2^\omega}$ are pairwise singular probability measures on $\{0,1\}^{\mathbb{N}}$, and $L^1(\lambda_\gamma) = L^1[0,1]$ or $L^1(\lambda_\gamma) = \mathbb{R}$.

Therefore,

$$\mu_{\xi,i} = \sum_{\gamma<2^\omega} \frac{d\mu_{\xi,i}}{d\lambda_\gamma}$$

where the sum is taken in ℓ^1-norm.

Choose a finite subset $F_{\xi,i}$ of 2^ω so that the measure

$$\mu'_{\xi,i} = \sum_{\gamma\in F_{\xi,i}} \frac{d\mu_{\xi,i}}{d\lambda_\gamma}$$

satisfies

$$||T\mu'_{\xi,i} - x^{**}_{\xi,i}|| < \frac{\delta}{256}. \tag{1}$$

In particular, for $\xi \neq \zeta$ we get

$$d(T\mu'_{\xi,i} - T\mu'_{\zeta,i}, E_u) > \frac{\delta}{4}. \tag{2}$$

Apply Erdös–Rado's Lemma ([C-N]) to the family $\{F_\xi = \bigcup_{i=1}^n F_{\xi,i},\ \xi < \omega_1\}$ to find an uncountable set $A \subset \omega_1$ and a finite set $F \subset 2^\omega$, such that for $\xi \neq \zeta$ in A

$$F_\xi \cap F_\zeta = F.$$

We set $\lambda_F = \sum_{\gamma\in F} \lambda_\gamma$ and for ξ in A

$$\nu_{\xi,i} = \mu'_{\xi,i} - \frac{d\mu'_{\xi,i}}{d\lambda_F}.$$

Claim: For all $i = 1,...,n$ the set $B_i = \{\xi \in A : d(T\nu_{\xi,i}, E_u) \leq \frac{\delta}{16}\}$ is at most countable.

To prove the claim suppose that for some i the set B_i is uncountable. Then, since $L^1(\lambda_F)$ is separable, there are $\xi \neq \zeta$ in B_i such that

$$||\frac{d\mu'_{\xi,i}}{d\lambda_F} - \frac{d\mu'_{\zeta,i}}{d\lambda_F}|| < \frac{\delta}{16}.$$

But then

$$d(T\mu'_{\xi,i} - T\mu'_{\zeta,i}, E_u) < \frac{\delta}{4}$$

which contradicts inequality (2); this completes the proof of the claim.

Choose $\xi_1 < \xi_2 < \cdots < \xi_n$ in A such that

$$d(T\nu_{\xi_i,i}, E_u) > \frac{\delta}{16}. \tag{3}$$

In the rest of the proof we shall denote (ξ_i, i) by ξ_i.

Notice that the measures $\nu_{\xi_1}, \ldots, \nu_{\xi_n}, \lambda_F$ are pairwise singular. Choose $U_1, \ldots, U_n$ pairwise disjoint clopen subsets of $\{0,1\}^{\mathbb{N}}$ such that for $i = 1, \ldots, n$

$$\|\nu_{\xi_i} \restriction U_i^C\| < \frac{\delta}{128} \quad \text{and} \quad \left\|\frac{d\mu_{\xi_i}}{d_{\lambda_F}} \restriction \bigcup_{j=1}^n U_j\right\| < \frac{\delta}{128}. \tag{4}$$

We are ready to prove the desired property. Indeed, for $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ we have

$$d\left(\sum_{i=1}^n \lambda_i T\mu'_{\xi_i}, E_u\right) \geq d\left(\sum_{i=1}^n \lambda_i T\mu'_{\xi_i} \restriction \bigcup_{j=1}^n U_j, E_u\right)$$

$$\geq d\left(\sum_{i=1}^n \lambda_i (T\nu_{\xi_i} \restriction U_i), E_u\right) - \sum_{i=1}^n \lambda_i \left\|T\nu_{\xi_i} \restriction \bigcup_{j\neq i} U_j\right\| - \sum_{i=1}^n \lambda_i \left\|\frac{d\mu_{\xi_i}}{d_{\lambda_F}} \restriction \bigcup_{j=1}^n U_j\right\|.$$

From Lemma 2 we get

$$d\left(\sum_{i=1}^n \lambda_i (T\nu_{\xi_i} \restriction U_i), E_u\right) \geq \frac{1}{2} \sum_{i=1}^n \lambda_i d(T\nu_{\xi_i} \restriction U_i, E_u)$$

and from (3) and (4) we get

$$d\left(\sum_{i=1}^n \lambda_i T\mu'_{\xi_i}, E_u\right) > \frac{1}{2}\frac{3\delta}{64} - \frac{\delta}{64} = \frac{\delta}{128}.$$

Finally, from (1) we have

$$d\left(\sum_{i=1}^n \lambda_i x^{**}_{\xi_i}, E_u\right) > \frac{\delta}{256}.$$

So (*) is proved and the proof of the Proposition is complete.

We note that our proof and P.4c yield, in fact, that C is not strongly regular. ∎

Remark: The space E does not contain a subspace isomorphic to $c_0(\mathbb{N})$. This is because $c_0(\mathbb{N})$ contains a non-RNP closed convex subset on which norm and weak topologies coincide. Since E fails the PCP and does not contain $c_0(\mathbb{N})$, it does not embed into a space with an unconditional skipped blocking finite dimensional decomposition. Finally, E semiembeds into E_u, a space with an unconditional basis.

PROPOSITION 5: *The properties RNP and KMP are equivalent on the subsets of E. Furthermore, if C is a closed convex non-RNP subset of E, then it contains a subset L with a $P\alpha\ell$-representation.*

Proof: As we mentioned before, if C is a closed convex bounded non-RNP set, then $J[C]$ carries the same properties and it is contained in E_u which has an unconditional basis. Therefore, there exists a closed convex subset L of $J[C]$ with a $P\alpha\ell$-representation [A-D]. Then $J^{-1}[L]$ has the same property. ∎

We conclude with the following result.

THEOREM 6: *Suppose that X is a separable Banach space such that X^{**}/X is isomorphic to $\ell^1(\Gamma)$. Then X has the RNP.*

Proof: Assume that X contains a δ-non-dentable subset C. Then the techniques developed in the proof of Proposition 4 show that C is not strongly regular. Actually, every convex combination $\sum_{i=1}^{n} \lambda_i S_i$ of slices of C will have diameter greater than $\delta/256$. Hence, by a result due to Bourgain [B], ℓ^1 embeds into X^*, and by Pelczynski's Theorem [P], $M[0,1]$ embeds into X^{**}. But then there exists a sequence $(x_n^{**})_{n\in\mathbb{N}}$ weakly convergent to zero with $d(x_n^{**}, X) > \varepsilon$, for some $\varepsilon > 0$. This contradicts the Schur property of $\ell^1(\Gamma)$. ∎

Remark: There are known results which show that for some sets Γ, $\ell^1(\Gamma)$ can be isomorphic to X^{**}/X for some separable space X. $\ell^1(N)$ has this property by a theorem of Lindenstrauss ([Li]). Odell in [O] has constructed a separable B-space X with $X^{**}/X \cong \ell^1(2^\omega)$.

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